香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

COMP 5212
Machine Learning
Lecture 4

Generalized Linear Models, Kernel Methods

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Feb 14, 2024

Recap

- Linear Regression $h_\theta(x) = \theta^T x$
- Logistic Regression $h_\theta(x) = g(\theta^T x)$
- Multi-class Classification Regression $h_\theta(x) = softmax(\theta_1^T x, \cdots, \theta_k^T x)$

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Is this coincidence?

Generalized Linear Models

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- We assume $y \mid x; \theta$ distributed as an exponential family.
  - Binary $\mapsto$ Bernoulli
  - Multiple Classses $\mapsto$ Multinomial
  - Real $\mapsto$ Gaussian
  - Counts $\mapsto$ Poisson
  - $\mathbb{R}_+$ $\mapsto$ Gamma, Exponential
  - Distributions $\mapsto$ Dirichlet

Exponential Family — Recap

Rough Idea *"If P has a a special form, then inference and learning come for free"*

$$P(y;\eta) = b(y)\exp\left\{\eta^T T(y) - a(\eta)\right\}.$$

η: natural parameter or canonical parameter

Here y, $a(\eta)$, and $b(y)$ are scalars. $T(y)$ same dimension as η.

$T(y)$ is called the **sufficient statistic**. holds all information the data provides with regard to the unknown parameter values

$b(y)$ is called the **base measure** – does *not* depend on η.

$a(\eta)$ is called the **log partition function** – does *not* depend on y.

$$1 = \sum_y P(y;\eta) = e^{-a(\eta)} \sum_y b(y)\exp\left\{\eta^T T(y)\right\}$$

$$\implies a(\eta) = \log \sum_y b(y)\exp\left\{\eta^T T(y)\right\}$$

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- Our model is *linear* beacuse we make the natural parameter $\eta = \theta^T x$ in which $\theta, x \in \mathbb{R}^{d+1}$.

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inference $h_\theta(x) = \mathbb{E}[y \mid x; \theta]$ is the **output**.

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$$\nabla_\eta a(\eta) = \frac{\sum_y T(y) b(y) \exp\{\eta^T T(y)\}}{\sum_y b(y) \exp\{\eta^T T(y)\}} = \mathbb{E}[T(y); \eta]$$

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$T(y) = y$ for most of the examples you will see in this course

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algorithm: SGD $\theta^{(t+1)} = \theta^{(t)} + \alpha \left(y^{(i)} - h_{\theta^{(t)}}(x^{(i)}) \right) x^{(i)}$

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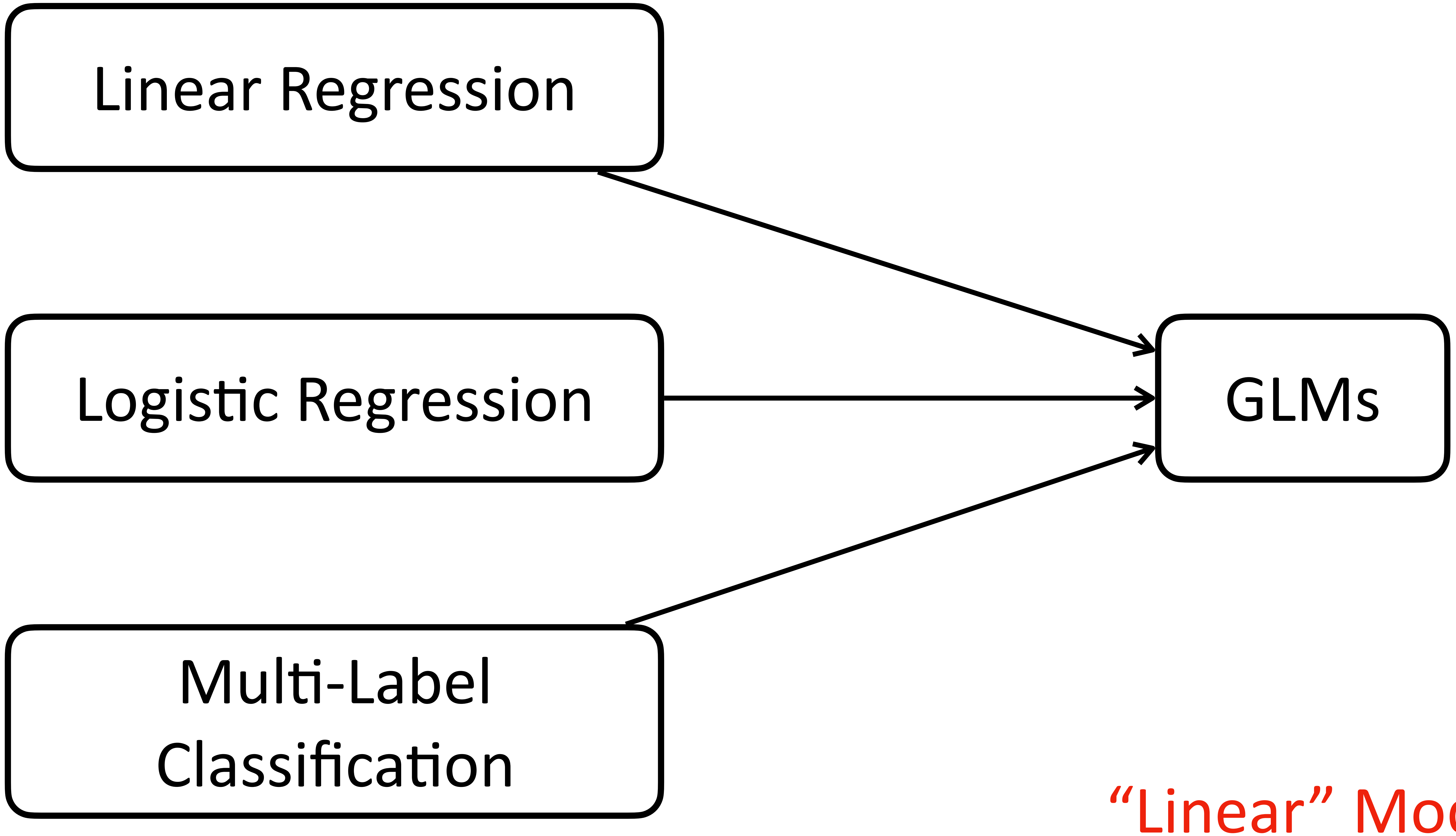
Enjoy closed-form solution for various statistics

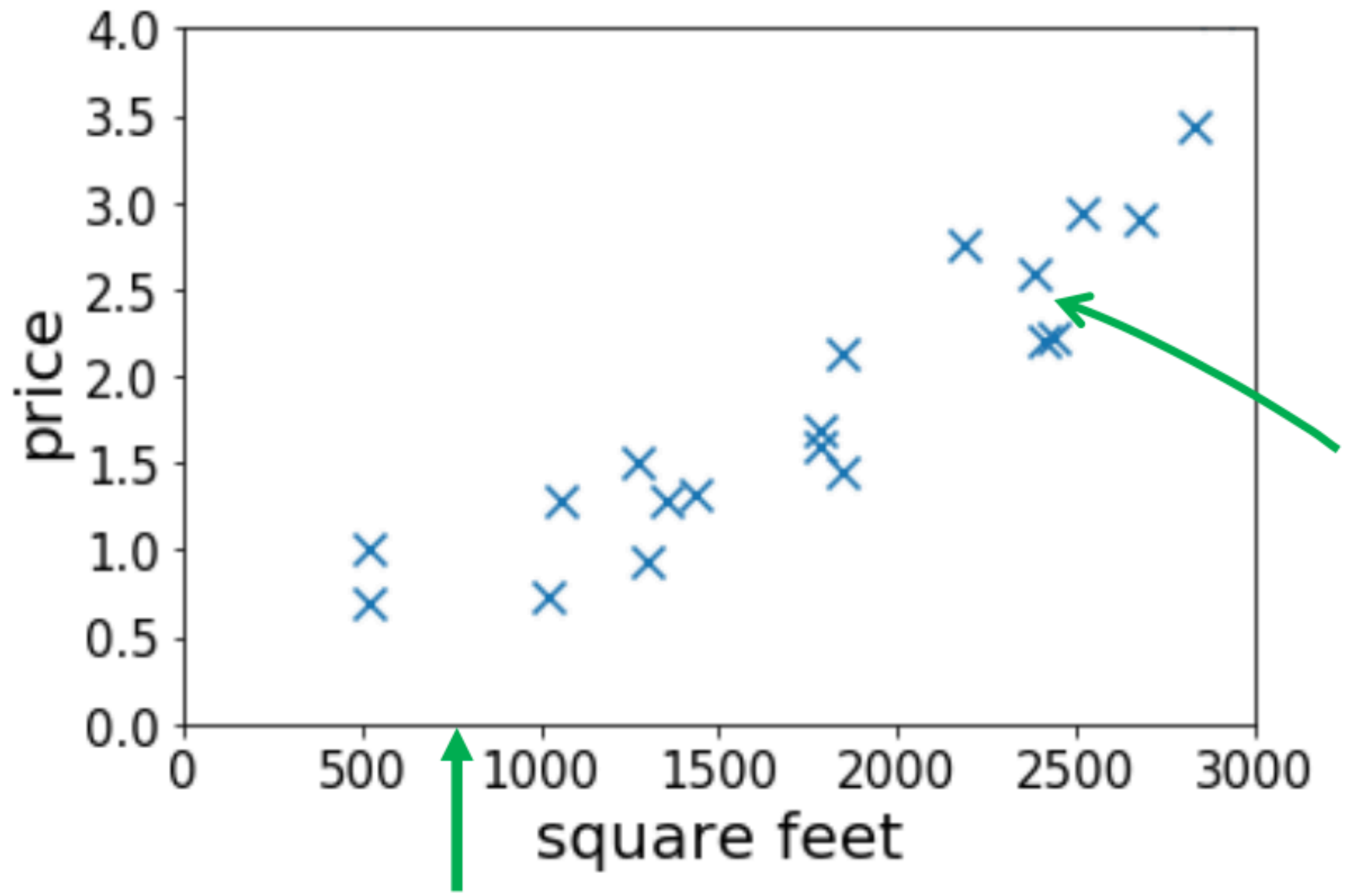
Generalized Linear Models



"Linear" Models

Kernel Methods

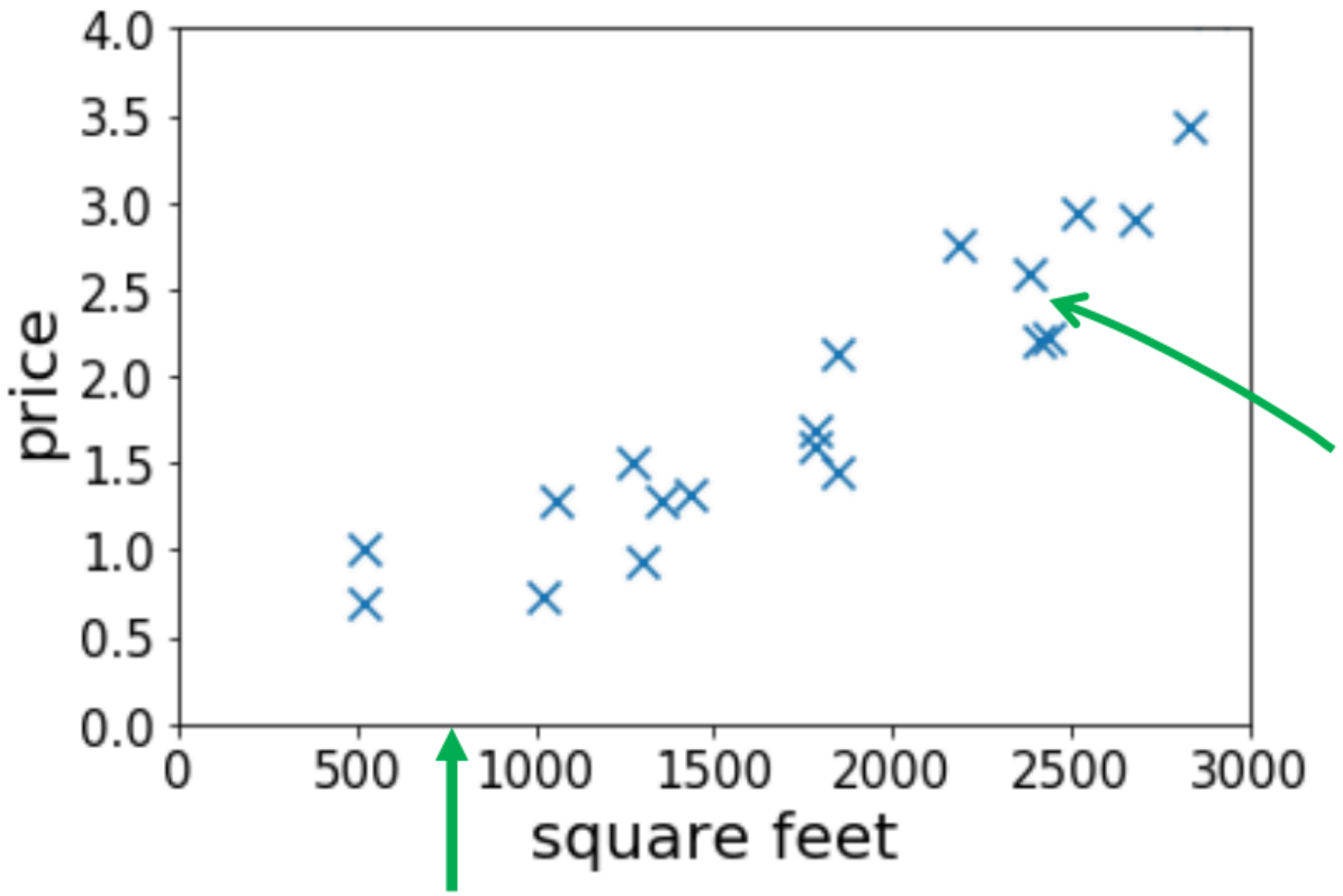
Feature Map



15th sample
$(x^{(15)}, y^{(15)})$

$x = 800$
$y = ?$

Feature Map



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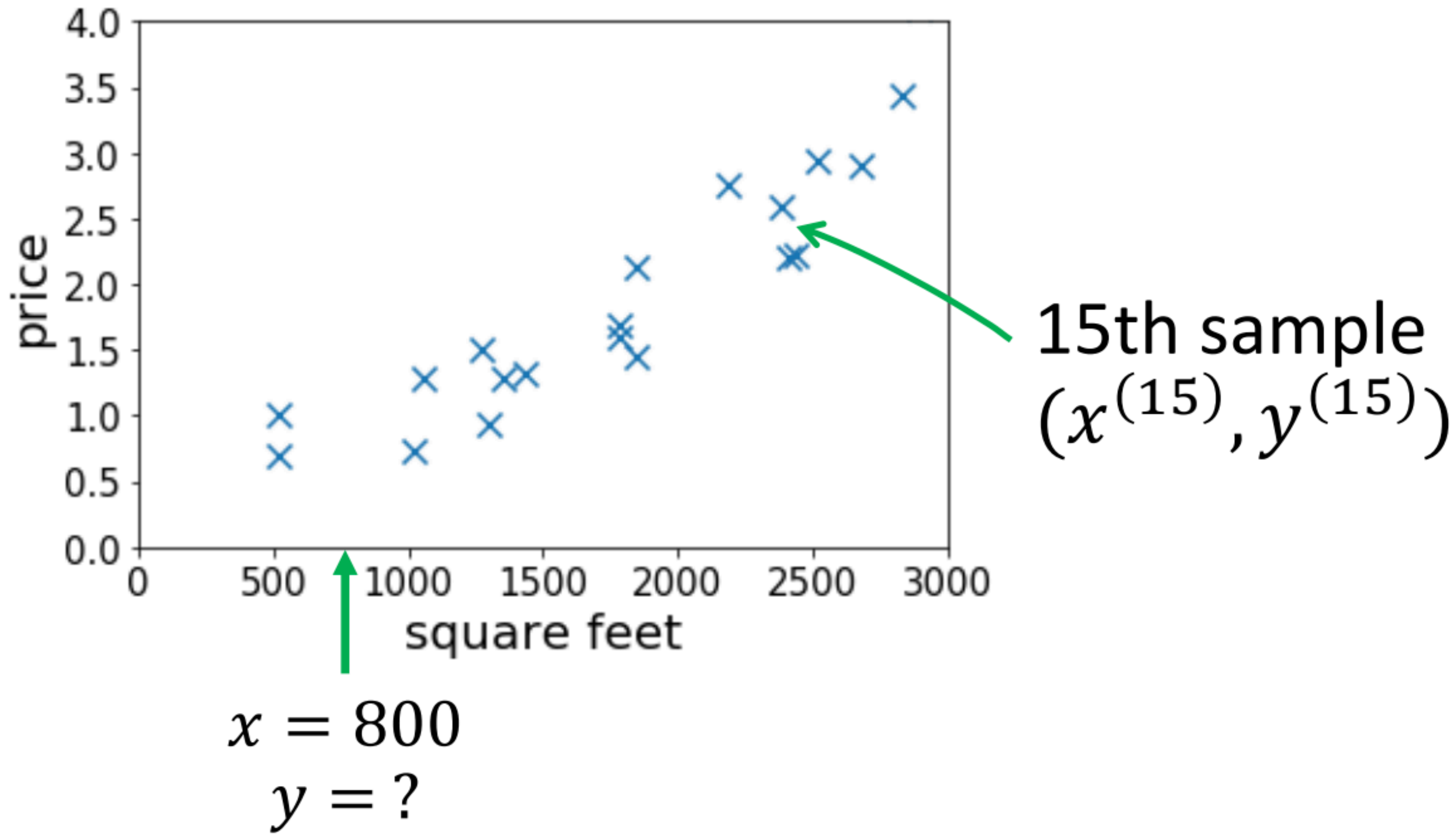

$$y = \theta x$$

$$y = \theta_3 x^3 + \theta_2 x^2 + \theta_1 x + \theta_0$$

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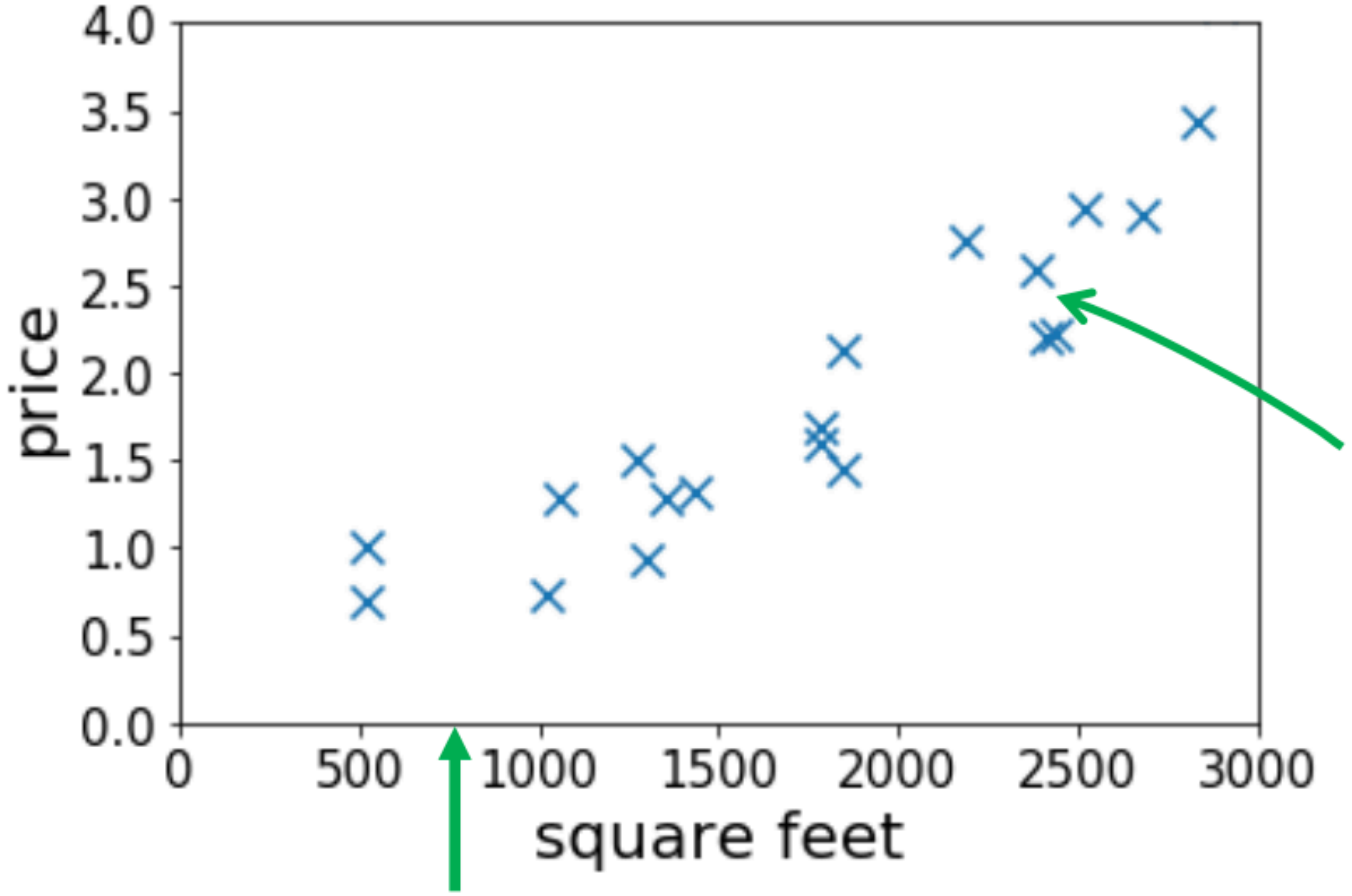

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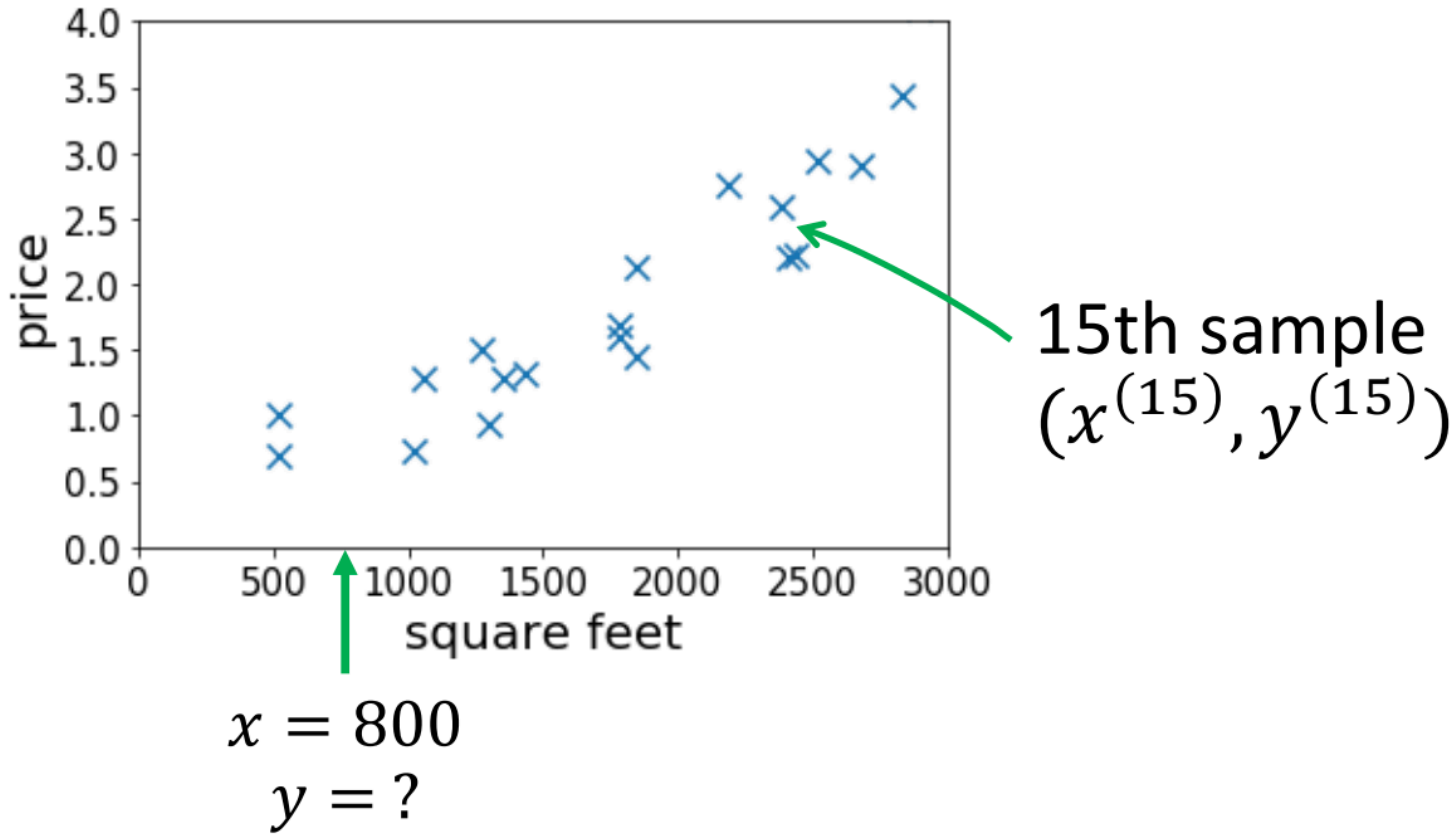

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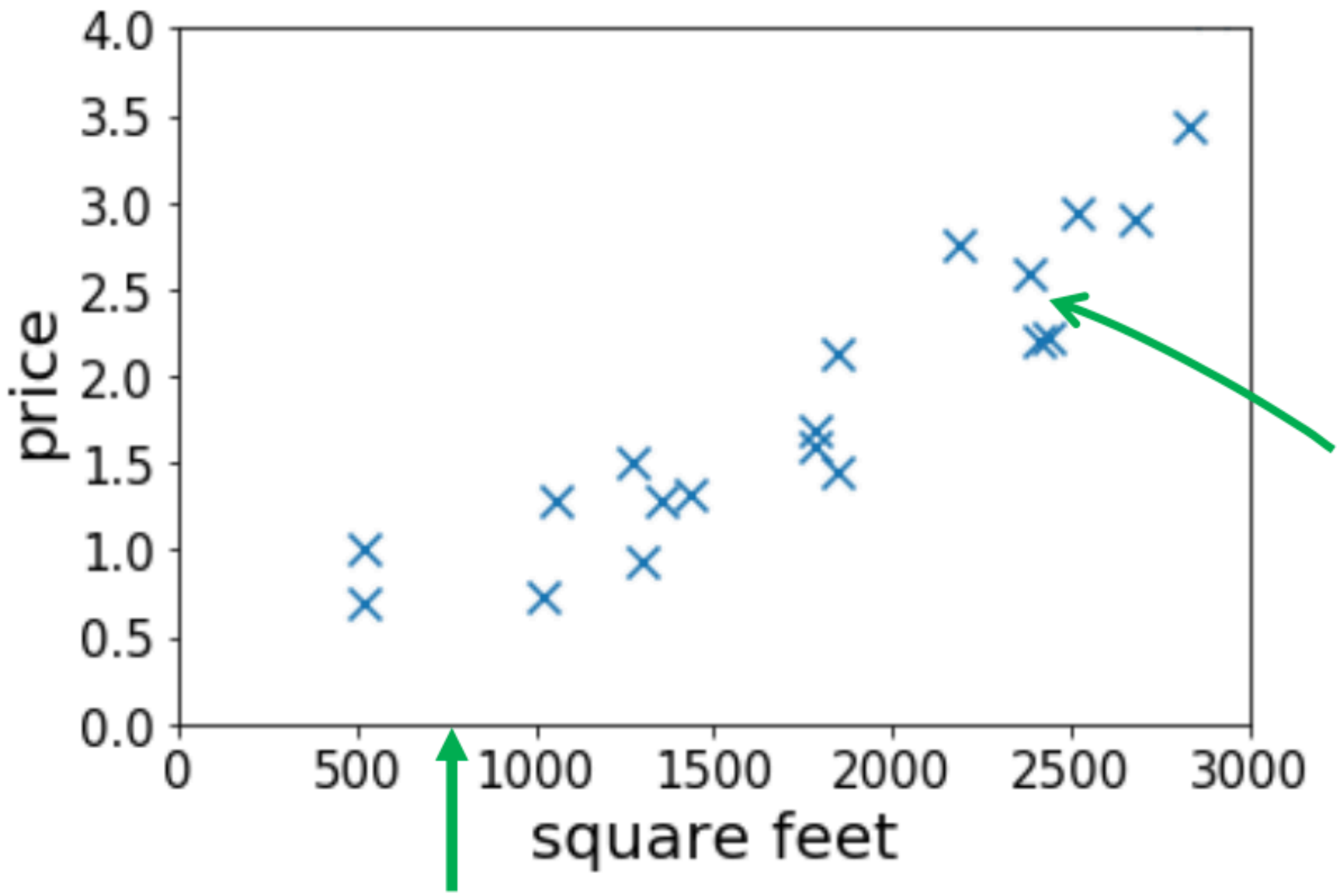

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Feature map
$\phi : R^d \rightarrow R^p$

$$y = \theta^T \phi(x)$$

LMS Update Rule with Features

Linear Regression:

$$\begin{aligned}\theta &:= \theta + \alpha \sum_{i=1}^{n} \left(y^{(i)} - h_\theta(x^{(i)})\right) x^{(i)} \\ &:= \theta + \alpha \sum_{i=1}^{n} \left(y^{(i)} - \theta^T x^{(i)}\right) x^{(i)}.\end{aligned}$$

With Features:

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How about Generalized Linear Models with Features?

New Feature Vector Can Be Very High-Dimensional

$$\phi(x) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_1^2 \\ x_1 x_2 \\ x_1 x_3 \\ \vdots \\ x_2 x_1 \\ \vdots \\ x_1^3 \\ x_1^2 x_2 \\ \vdots \end{bmatrix}$$

Computationally expensive

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Computationally expensive

Is the computation evitable given $\theta \in R^p$?

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We can precompute all pairwise $< \phi(x^{(j)}), \phi(x^{(i)}) >$ beforehand, and reuse it for every gradient descent update

Kernel Trick

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Kernel $K(x, z)$ $\quad \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ $\quad \mathcal{X}$ is the space of the input

$$K(x, z) \triangleq \langle \phi(x), \phi(z) \rangle$$

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Recall that n is the number of data samples

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Or in vector notation, letting K be the $n \times n$ matrix with $K_{ij} = K(x^{(i)}, x^{(j)})$, we have

$$\beta := \beta + \alpha(\vec{y} - K\beta)$$

Inference

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$$\theta^T \phi(x) = \sum_{i=1}^{n} \beta_i \phi(x^{(i)})^T \phi(x) = \sum_{i=1}^{n} \beta_i K(x^{(i)}, x)$$

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The Kernel function is all we need for training and inference!

Implicit Feature Map

Do we still need to define feature maps?

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What kinds of kernel functions K() can correspond to some feature map ϕ

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$$K(x, z) = (x^T z)^2 \qquad x, z \in \mathbb{R}^d$$

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Requires O(d) compute for Kernel function

Next Lecture

- What kinds of functions would make a kernel function?
- Infinite dimensions of feature mapping?
- Support Vector Machines

Thank You!
Q & A