香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

COMP 5212
Machine Learning
Lecture 6

Support Vector Machines

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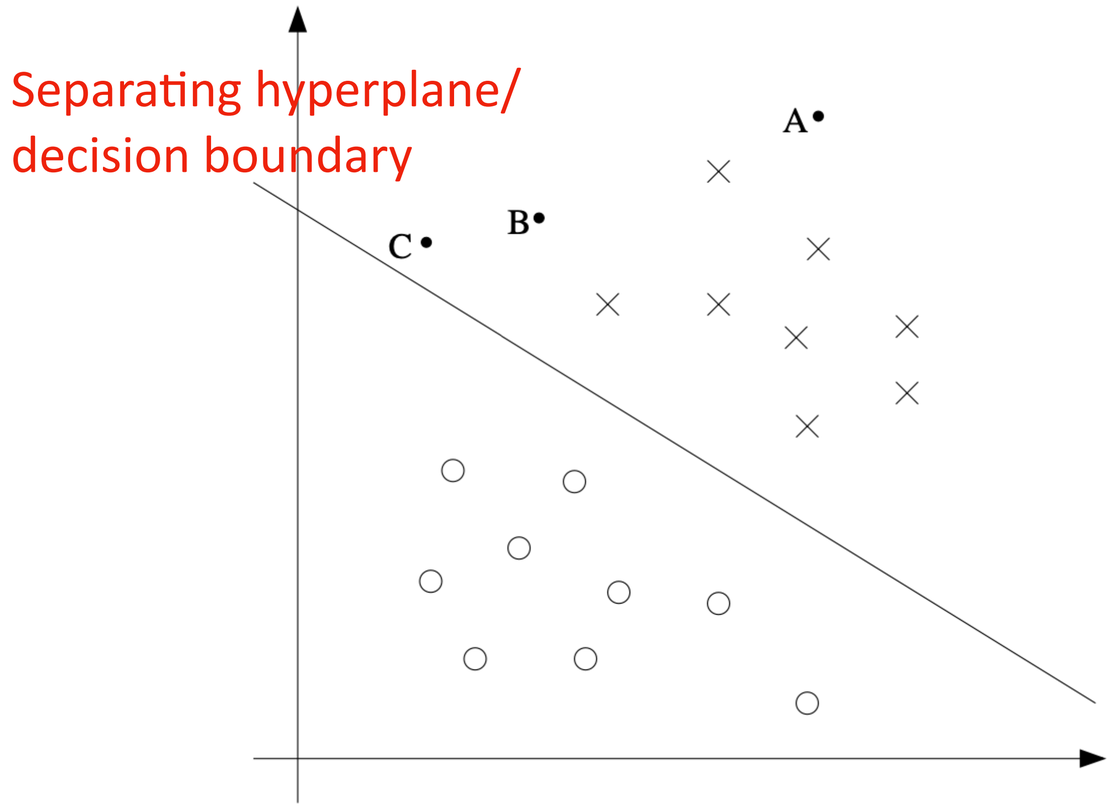
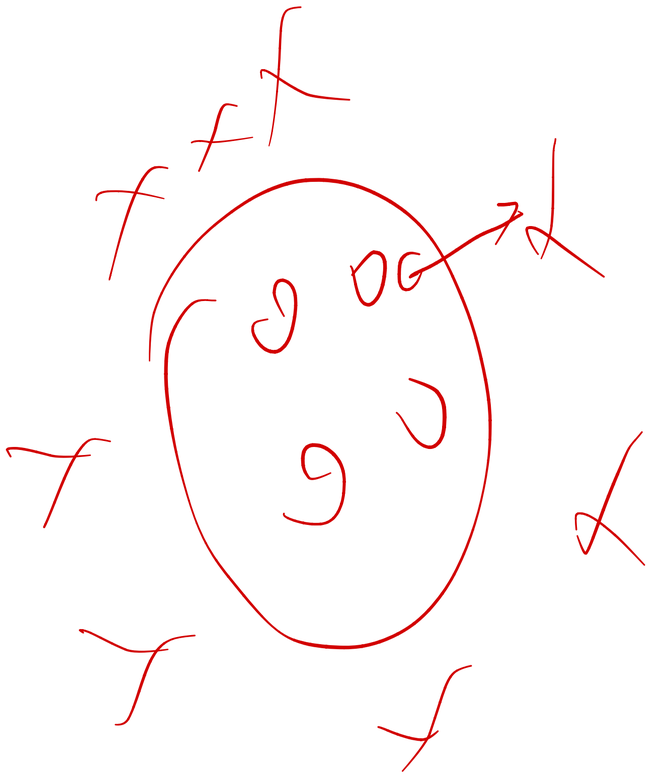
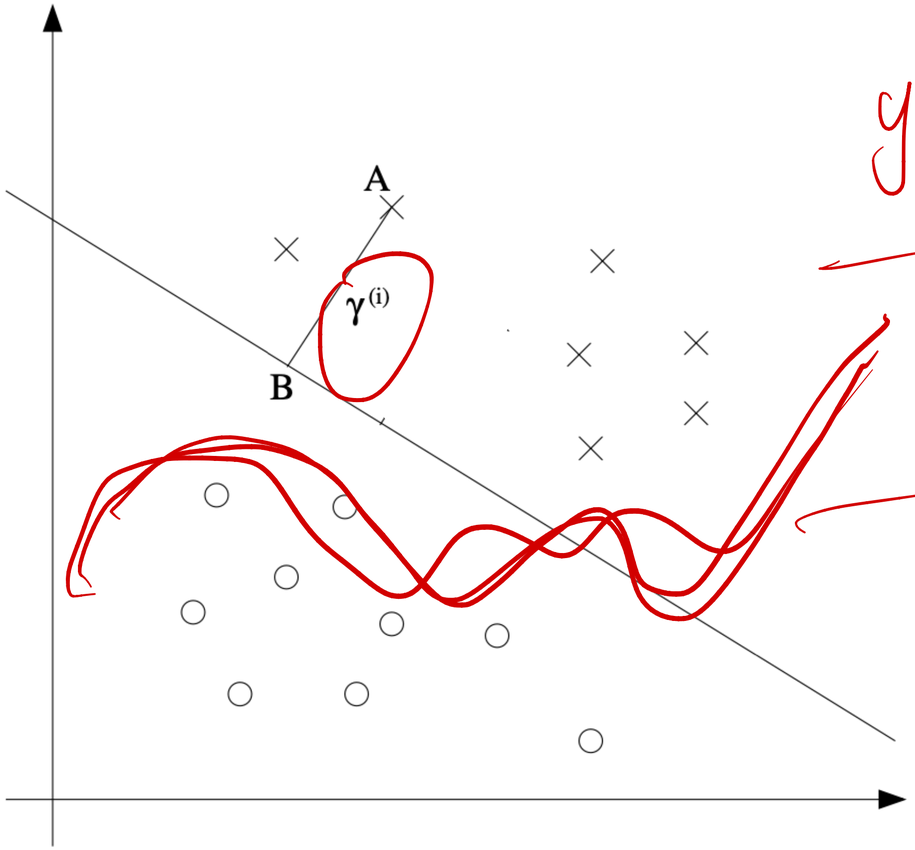
Mar 3, 2025

Recap: Support Vector Machines



Recap: Geometric Margin





What is the geometric margin?

Recap: Functional Margin

Given a training example $(x^{(i)}, y^{(i)})$

$$\hat{\gamma}^{(i)} = y^{(i)}(w^T x^{(i)} + b).$$

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$\frac{\hat{\gamma}}{\|w\|} = \gamma$

Given a training set $S = \{(x^{(i)}, y^{(i)}); i = 1,\ldots,n\}$

$$\hat{\gamma} = \min_{i=1,\ldots,n} \hat{\gamma}^{(i)}$$

Recap: Functional Margin

Given a training example $(x^{(i)}, y^{(i)})$

$$\hat{\gamma}^{(i)} = y^{(i)}(w^T x^{(i)} + b).$$

$w \to 2w$

$b \to 2b$

Given a training set $S = \{(x^{(i)}, y^{(i)}); i = 1,\ldots,n\}$

$$\hat{\gamma} = \min_{i=1,\ldots,n} \hat{\gamma}^{(i)}$$

$w^T x + b = 0$

Functional margin changes rescaling parameters, making it a bad objective, e.g. when w->2w, b->2b, the functional margin changes while the separating plane does not really change

Recap: Geometric Margin

Given a training set $S = \{(x^{(i)}, y^{(i)}); i = 1,\ldots,n\}$

$$\gamma = \min_{i=1,\ldots,n} \gamma^{(i)}$$

max min $\gamma^{(i)}$
γ $i \in 1,\ldots,n$

Recap: The Optimization Problem

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Infinite solutions, as $\hat{\gamma}$ can be at any scale without changing the classifier

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||w|| is not easy to deal with, non-convex objective

Recap: The Optimization Problem

$$\max_{w,b} \min_{i=1,\ldots,n} \gamma^{(i)}$$

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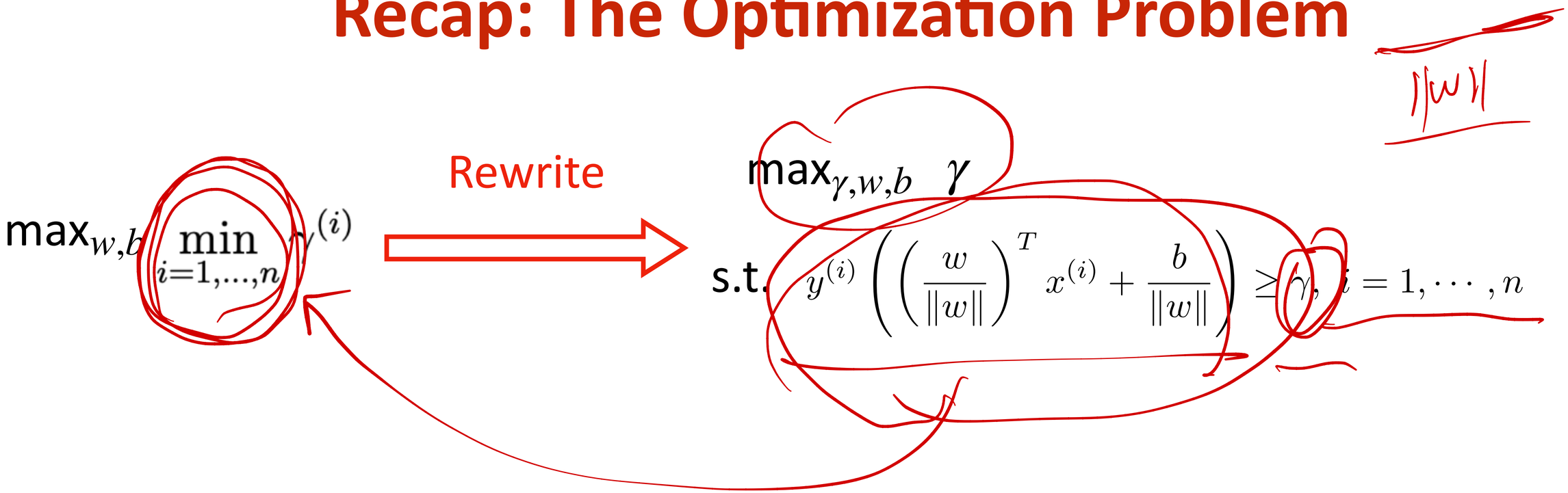

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Recap: The Optimization Problem

$\max_{w,b} \min_{i=1,\ldots,n} \gamma^{(i)}$

Rewrite

$\max_{\gamma,w,b} \ \gamma$

s.t. $y^{(i)}\left(\left(\frac{w}{\|w\|}\right)^T x^{(i)} + \frac{b}{\|w\|}\right) \geq \gamma, \ i = 1, \cdots, n$

Linear constraint

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$\max_{\hat{\gamma},w,b} \ \frac{\hat{\gamma}}{\|w\|}$

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Add constraint $\hat{\gamma} = 1$

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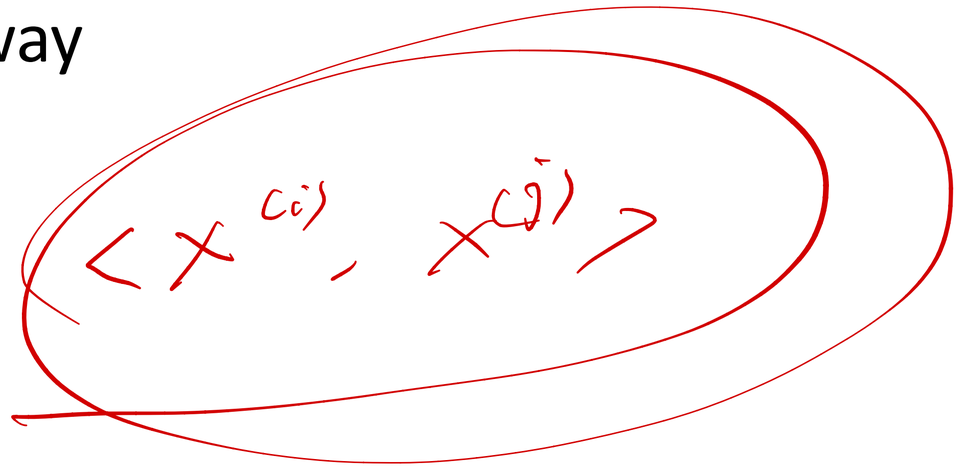
Assumption: the training dataset is linearly separable

The Dual Problem in Optimization

In optimization, sometimes the primal optimization is hard to solve, then we may find a related alternative optimization problem that can be solved more easily, to solve the orignal problem in an indirect way



Quadratic Program

Quadratic Program

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This is already a standard convex opt problem that is ready to be solved, why are we doing all the rest of things?

Generalized Lagrangian

Generalized Lagrangian

Primal optimization problem

$$\begin{aligned} \min_w \quad & f(w) \\ \text{s.t.} \quad & g_i(w) \leq 0, \quad i = 1, \ldots, k \\ & h_i(w) = 0, \quad i = 1, \ldots, l \end{aligned}$$

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Generalized Lagrangian

$$\mathcal{L}(w, \alpha, \beta) = f(w) + \sum_{i=1}^{k} \alpha_i g_i(w) + \sum_{i=1}^{l} \beta_i h_i(w)$$

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$$\theta_{\mathcal{P}}(w) = \max_{\alpha, \beta : \alpha_i \geq 0} f(w) + \sum_{i=1}^{k} \alpha_i g_i(w) + \sum_{i=1}^{l} \beta_i h_i(w)$$

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$$\theta_{\mathcal{P}}(w) = \begin{cases} f(w) & \text{if } w \text{ satisfies primal constraints} \\ \infty & \text{otherwise.} \end{cases}$$

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The primal optimization problem

$$\min_{w} \theta_{\mathcal{P}}(w) = \min_{w} \max_{\alpha,\beta\,:\,\alpha_i \geq 0} \mathcal{L}(w, \alpha, \beta)$$

The Dual Problem

The Dual Problem

Lagrangian function

$$\mathcal{L}(w, b, \alpha) = \frac{1}{2}||w||^2 - \sum_{i=1}^{n} \alpha_i \left[y^{(i)}(w^T x^{(i)} + b) - 1 \right]$$

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$$\theta(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} y^{(i)} y^{(j)} \alpha_i \alpha_j (x^{(i)})^T x^{(j)}$$

dual objective

The Dual Problem

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$$\text{s.t.} \quad \alpha_i \geq 0, \quad i = 1, \ldots, n$$

kernel

convex quadratic programm

$K(x^{(i)}, x^{(j)})$

The Dual Problem

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s.t. $\alpha_i \geq 0, \quad i = 1, \ldots, n$

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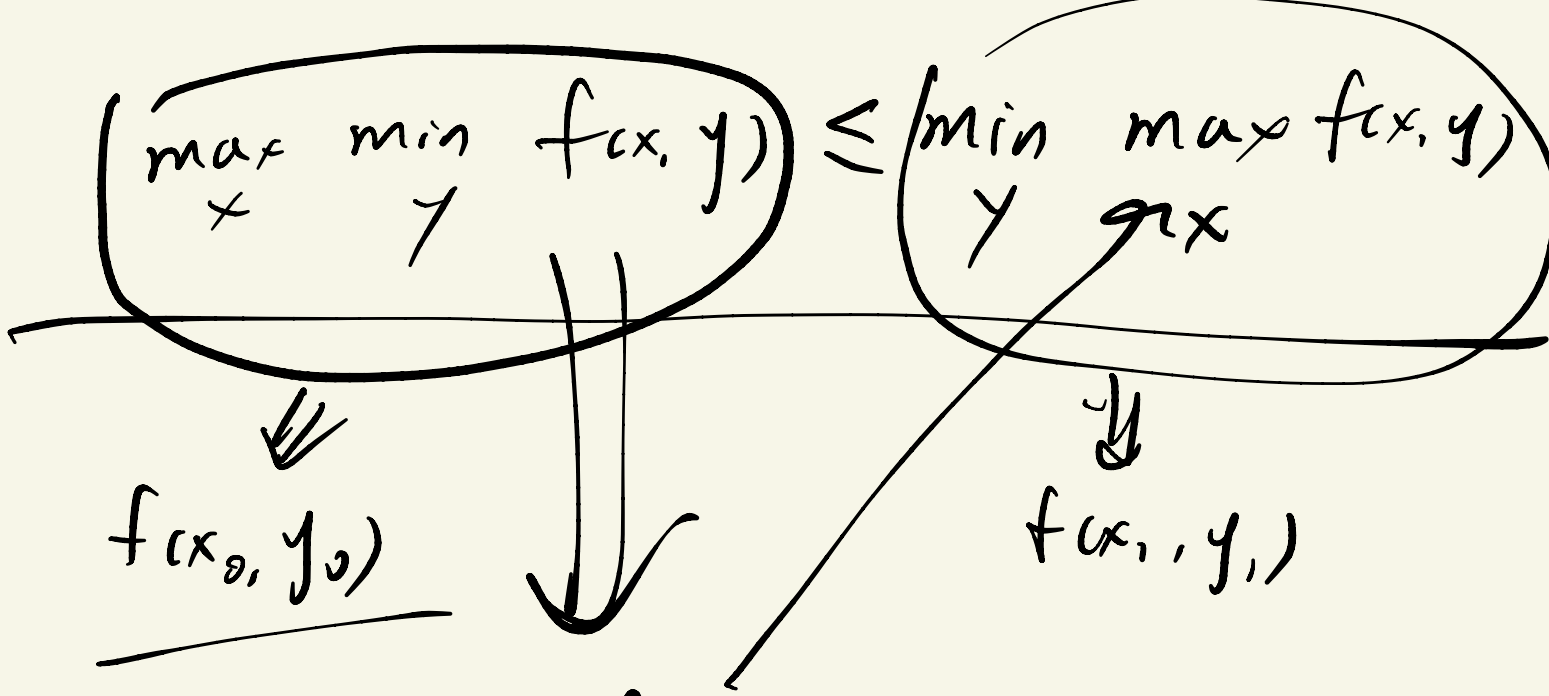
$$\frac{\partial}{\partial b} \mathcal{L}(w, b, \alpha) = \sum_{i=1}^{n} \alpha_i y^{(i)} = 0$$

What is the relation between solving this dual problem and solving the original problem

The Dual Problem

$$d^* = \max_{\alpha,\beta\,:\,\alpha_i \geq 0} \min_{w} \mathcal{L}(w, \alpha, \beta) \leq \min_{w} \max_{\alpha,\beta\,:\,\alpha_i \geq 0} \mathcal{L}(w, \alpha, \beta) = p^*$$

$d^* \leq p^*$



$f(x_1, y_0) \le f(x_2, y)$ for any x, y

$f(x_1, y) \ge f(x_2, y)$ for any x, y

$f(x_0, y_0) \le f(x_1, y_0) \le f(x_1, y_1)$

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$$\max_{x} \min_{y} f(x,y) \leq \min_{y} \max_{x} f(x,y)$$

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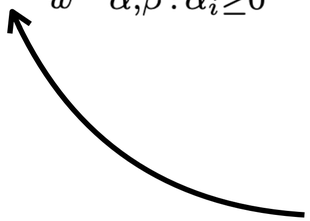

$$\max_x \min_y f(x,y) \leq \min_y \max_x f(x,y)$$

Under certain conditions: $d^* = p^*$

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Under certain conditions: $d^* = p^*$ Zero-duality Gap

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Under certain conditions: $d^* = p^*$ Zero-duality Gap

What are the conditions?

Slater's Condition

$$\begin{aligned} \min_w \quad & f(w) \\ \text{s.t.} \quad & g_i(w) \leq 0, \quad i = 1, \ldots, k \\ & h_i(w) = 0, \quad i = 1, \ldots, l. \end{aligned}$$

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- f(w) and g(w) are convex
- $h_i(w)$ is affine (i.e. linear)
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If slater's condition holds, then $d^* = p^*$

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The primal optimization problem of SVM satisfies the slater's condition

KKT Conditions

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$$\begin{aligned}
\frac{\partial}{\partial w_i} \mathcal{L}(w^*, \alpha^*, \beta^*) &= 0, \quad i = 1, \ldots, d \\
\frac{\partial}{\partial \beta_i} \mathcal{L}(w^*, \alpha^*, \beta^*) &= 0, \quad i = 1, \ldots, l \\
\alpha_i^* g_i(w^*) &= 0, \quad i = 1, \ldots, k \\
g_i(w^*) &\leq 0, \quad i = 1, \ldots, k \\
\alpha^* &\geq 0, \quad i = 1, \ldots, k
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\alpha_i^* g_i(w^*) &= 0, \quad i = 1, \ldots, k \\
g_i(w^*) &\leq 0, \quad i = 1, \ldots, k \\
\alpha^* &\geq 0, \quad i = 1, \ldots, k
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Normal Lagrange multiplier equations

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\frac{\partial}{\partial \beta_i} \mathcal{L}(w^*, \alpha^*, \beta^*) &= 0, \quad i = 1, \ldots, l \\
\alpha_i^* g_i(w^*) &= 0, \quad i = 1, \ldots, k \\
g_i(w^*) &\leq 0, \quad i = 1, \ldots, k \\
\alpha^* &\geq 0, \quad i = 1, \ldots, k
\end{aligned}$$

Normal Lagrange multiplier equations

The original constraints

KKT Conditions

Denote the solution to the primal problem as w^*, the solution to the dual problem as α^*, β^*, then zero duality gap is sufficient and necessary (i.e. equivalent) to satisfy KKT Conditions:

$$\mathcal{L}(w, \alpha, \beta) = f(w) + \sum_{i=1}^{k} \alpha_i g_i(w) + \sum_{i=1}^{l} \beta_i h_i(w)$$

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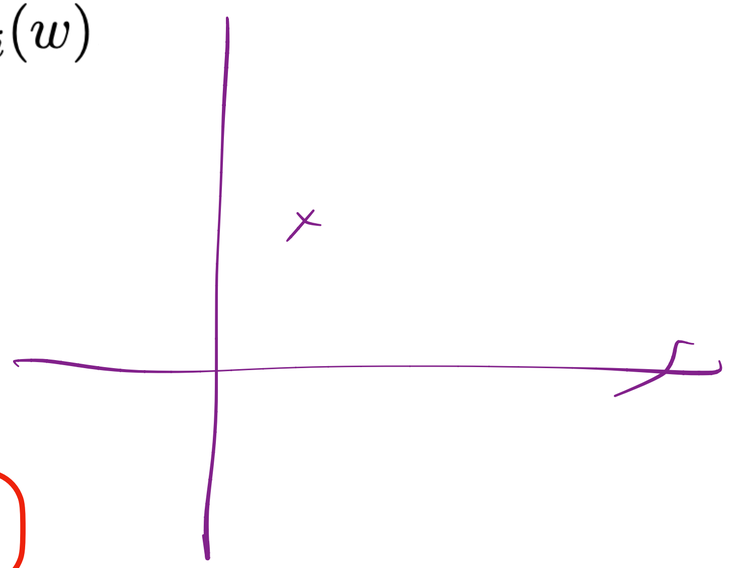
$\alpha \geq 0$

α_i

If $\alpha_i^* > 0$, then $g_i(w^*) = 0$, the inequality is actually equality

$y(w^T x + b) = 1$



Supporting Vectors

Supporting Vectors

$$\alpha_i^* g_i(w^*) \quad = \quad 0, \quad i = 1, \ldots, k$$

Supporting Vectors

$$\alpha_i^* g_i(w^*) = 0, \quad i = 1, \ldots, k$$

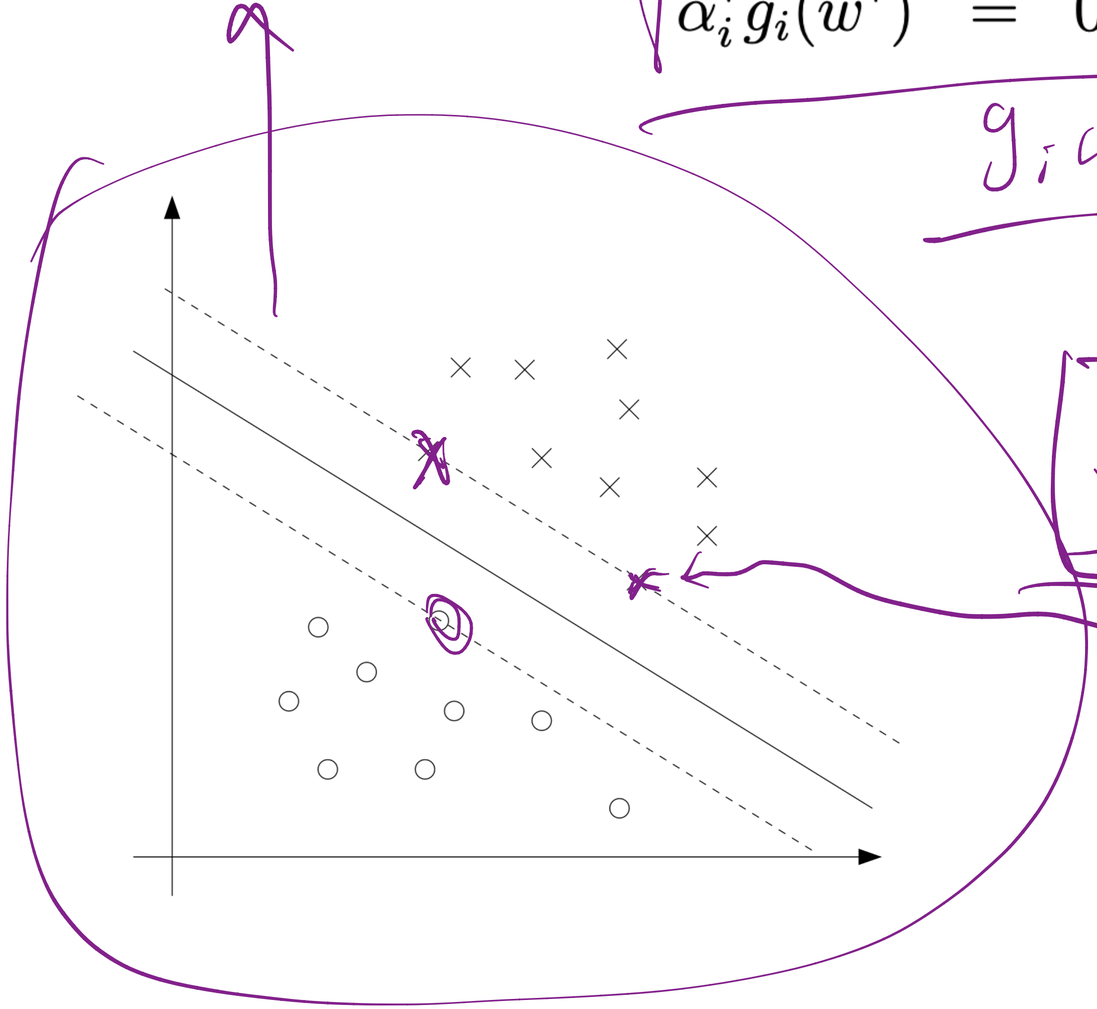
$g_i(w^*) = 0$

$y^{(i)}(w^T x^{(i)} + b) = 1$

$\gamma = 1$

$\gamma = \min_{i=1,2,\ldots,n} \hat{\gamma}^{(i)}$

$\hat{\gamma}^{(i)}$



Supporting Vectors

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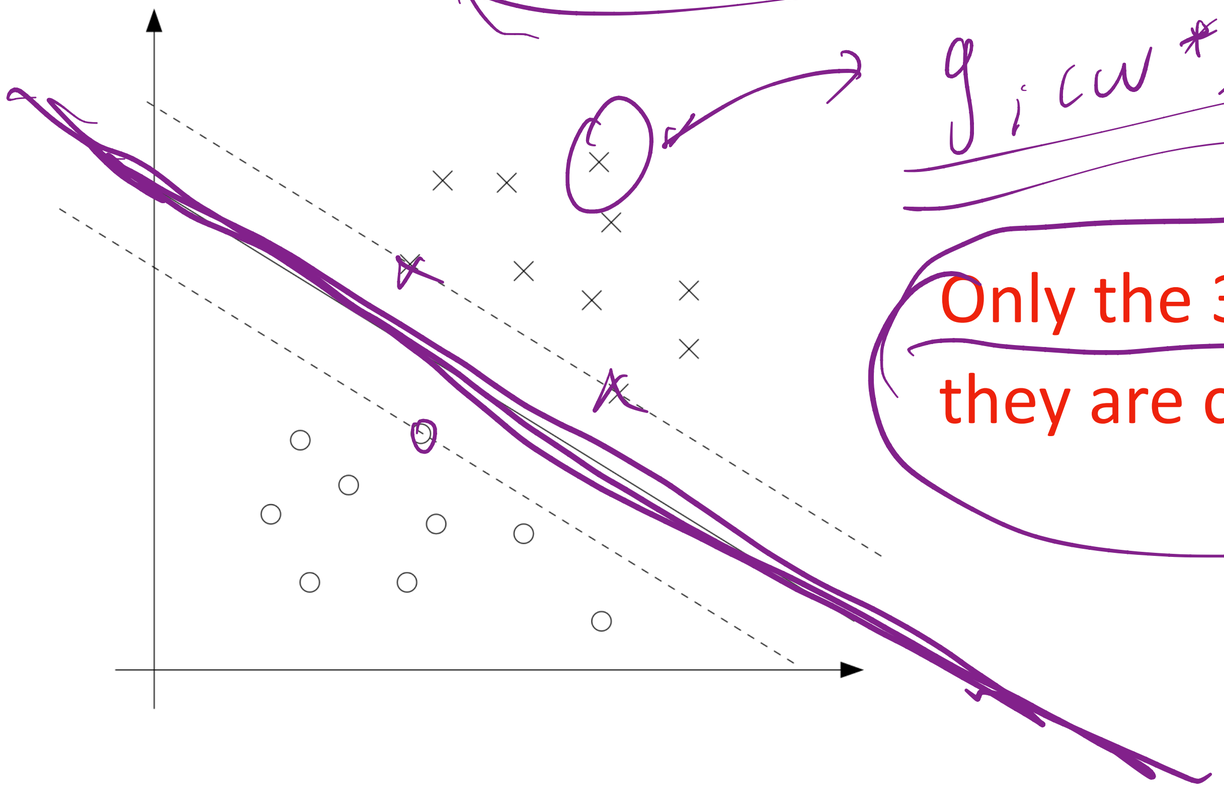

Only the 3 points have non-zero α_i, and they are called supporting vectors

The Dual Problem of SVM

The Dual Problem of SVM

$$\max_\alpha \quad W(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} y^{(i)} y^{(j)} \alpha_i \alpha_j \langle x^{(i)}, x^{(j)} \rangle$$

$$\text{s.t.} \quad \alpha_i \geq 0, \quad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} \alpha_i y^{(i)} = 0,$$

linear

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Kernel is all we need!

α_i, α_j

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After solving α (we'll talk about how later)

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From the original constraints

Inference

Inference

$$w^T x + b = \left(\sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)} \right)^T x + b$$

$$= \sum_{i=1}^{n} \alpha_i y^{(i)} \langle x^{(i)}, x \rangle + b.$$

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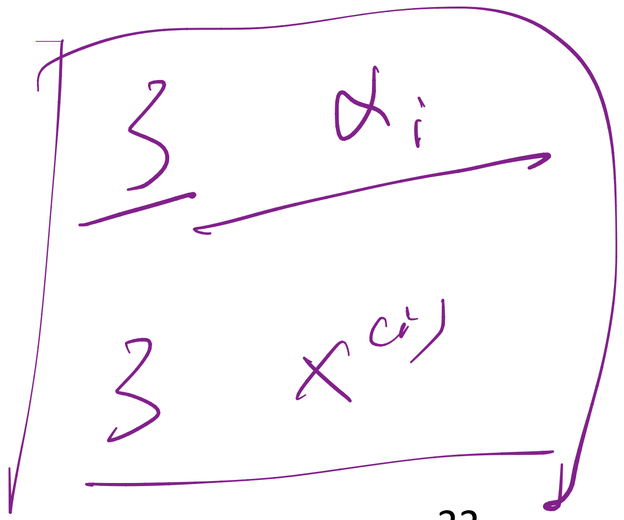
Supporting vector

kernel

We never need to really compute w



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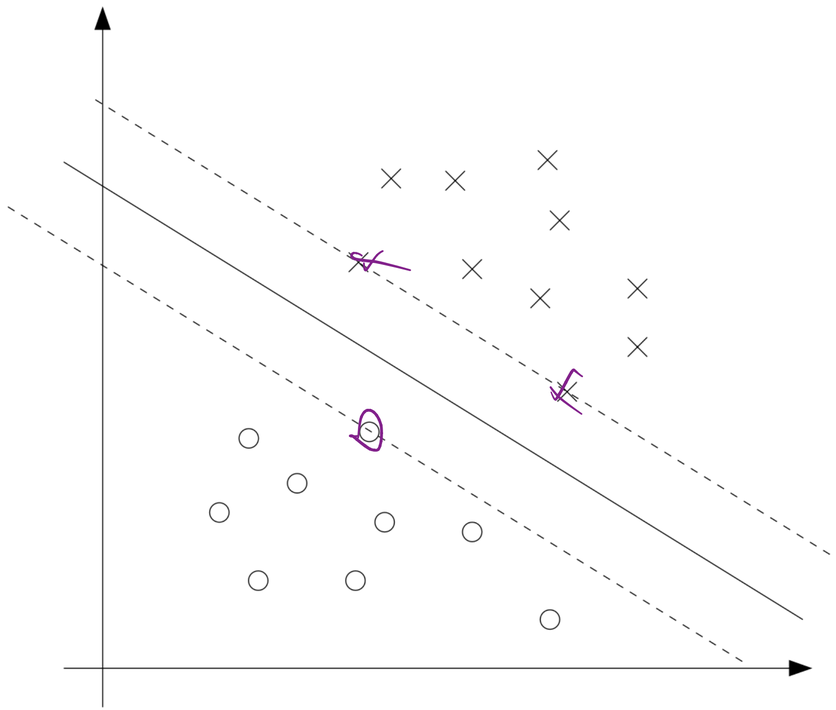

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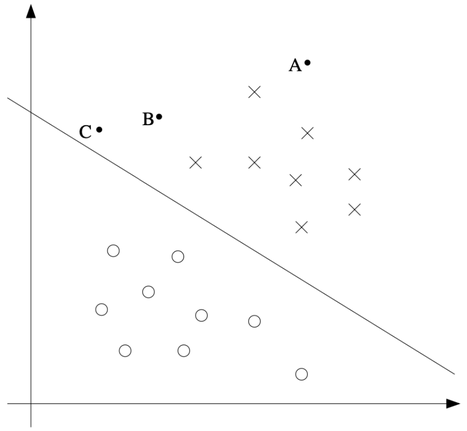
Review of the High-Level Logic



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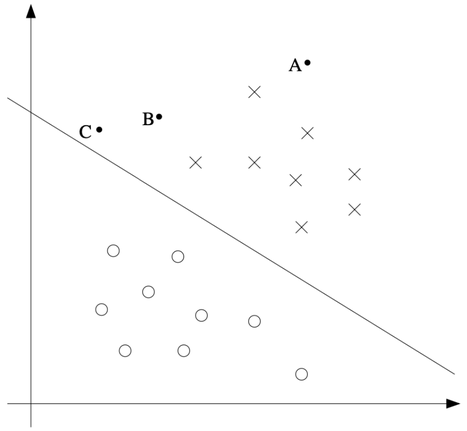
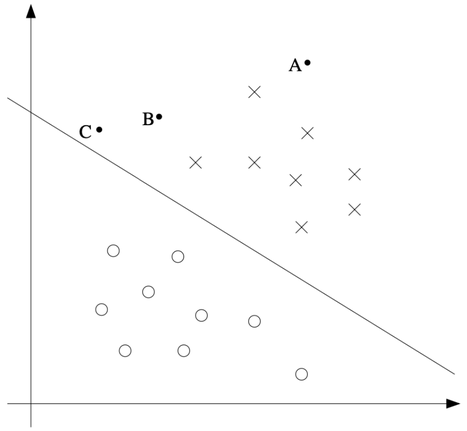


Review of the High-Level Logic

$$h_{w,b}(x) = g(w^T x + b)$$



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Maximize geometric margin

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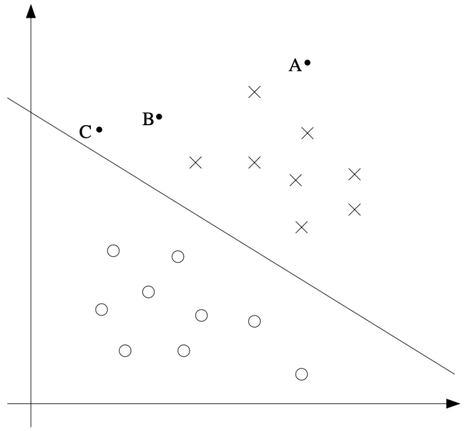

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Problem rewriting →

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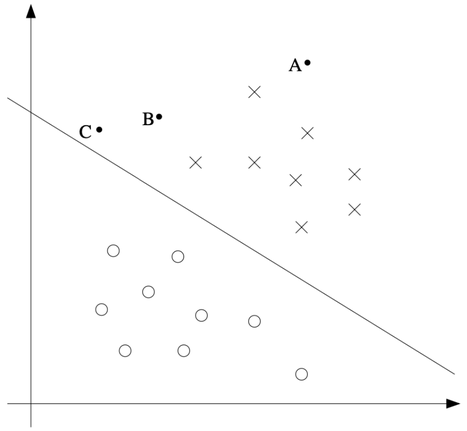
∇

$||w||$



Review of the High-Level Logic

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Problem rewriting

Quadratic Optimization Problem

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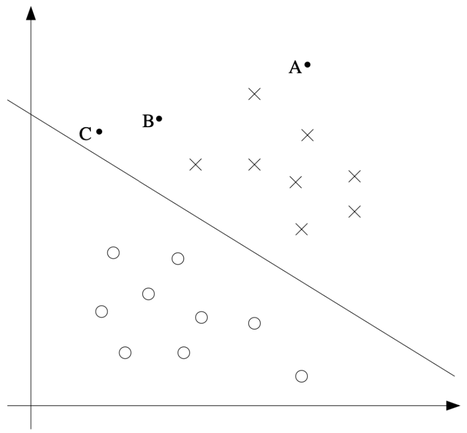

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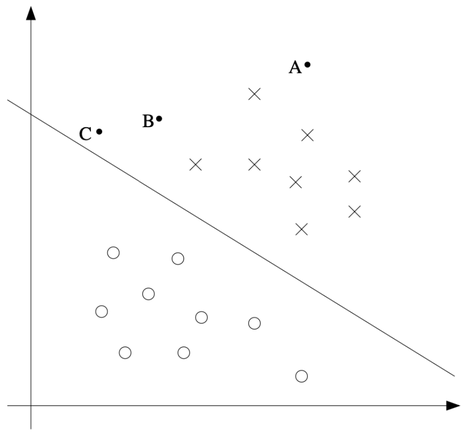
Not suitable for non-linear cases (high-dim feature map)



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Finding a related optimization problem that is easier

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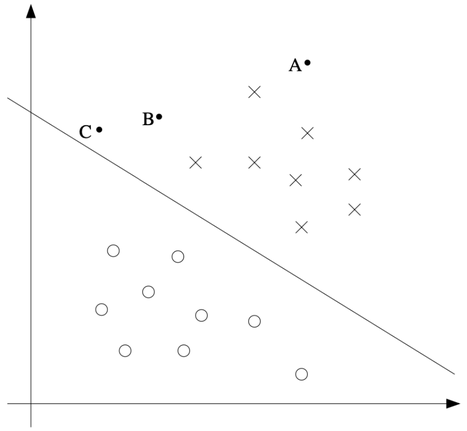

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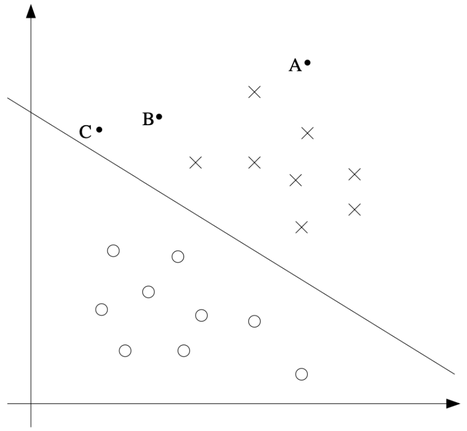

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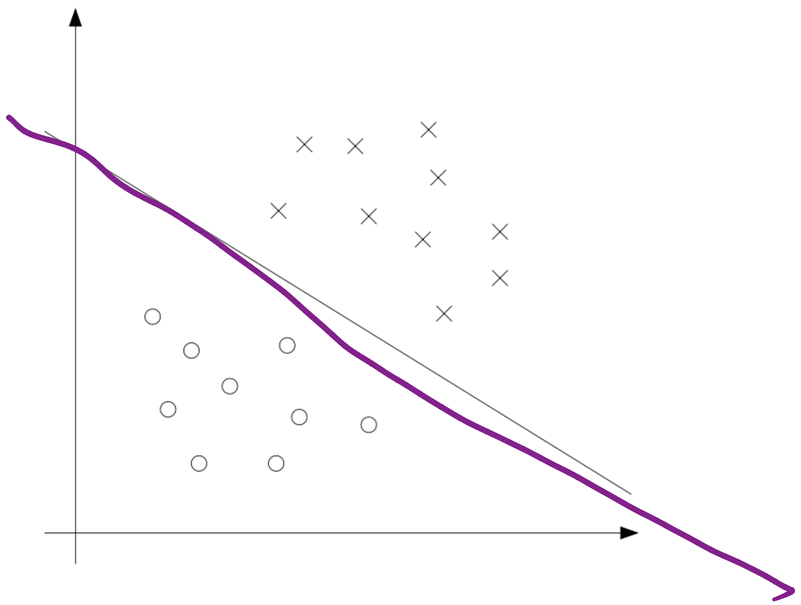
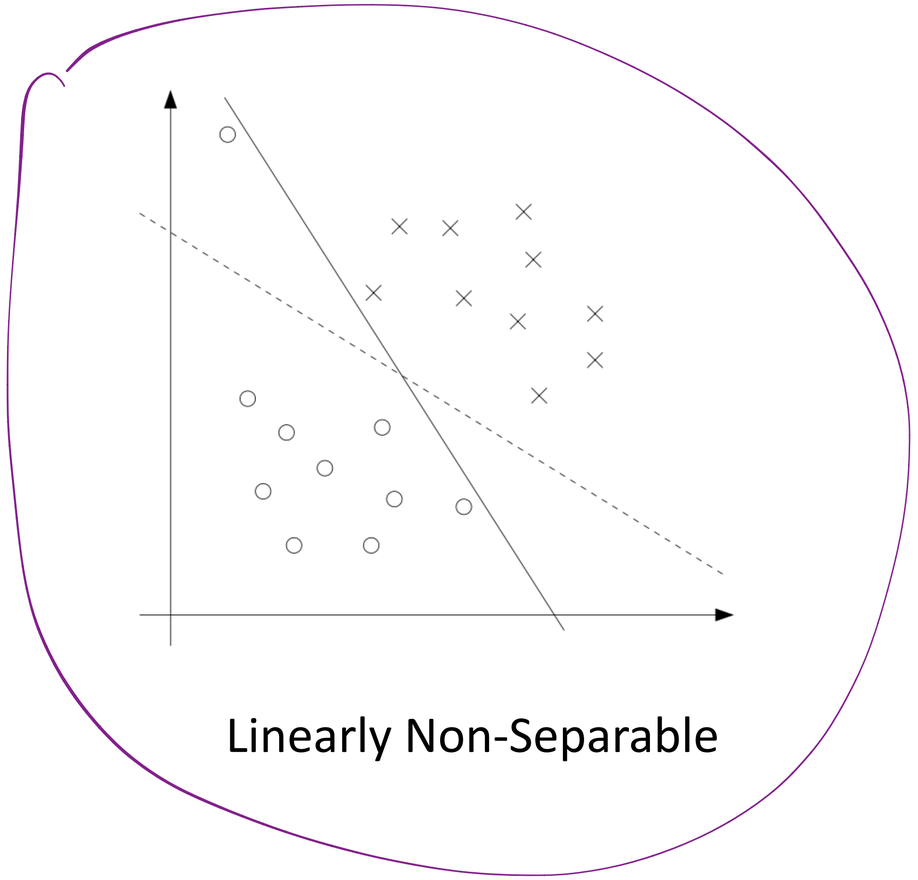
Kernel makes it very flexible in non-linear cases!

The Non-Separable Case



Linearly Separable



Linearly Non-Separable

$y^{(i)}(w^T x + b) \geq 1$

The Non-Separable Case

Primal opt problem:

$$\min_{\gamma,w,b} \quad \frac{1}{2}||w||^2 + C\sum_{i=1}^{n}\xi_i$$

$$\text{s.t.} \quad y^{(i)}(w^T x^{(i)} + b) \geq 1 - \xi_i, \quad i = 1, \ldots, n$$

$$\xi_i \geq 0, \quad i = 1, \ldots, n.$$

Dual opt problem

$$\max_{\alpha} \quad W(\alpha) = \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i,j=1}^{n} y^{(i)}y^{(j)}\alpha_i\alpha_j\langle x^{(i)}, x^{(j)}\rangle$$

$$\text{s.t.} \quad 0 \leq \alpha_i \leq C, \quad i = 1, \ldots, n$$

$$\sum_{i=1}^{n}\alpha_i y^{(i)} = 0,$$

Thank You!
Q & A